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Some results on characterizing the generalized cyclic designs

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SUMMARY

In this note a new characterization of generalized cyclic designs is derived by using an idea of inter-contrasts. The final section gives an insight to the fields of application.

KEY WORDS: circulant matrix, crop rotation experiment, GCIB design, inter-contrast.

1. Introduction

The contribution we make in this note is in the following design context. Suppose there are given $t = mn$ treatments labelled $0, 1, \ldots, t-1$ to be compared via n blocks of size k each ($k \leq t$). Let the treatments be arranged according to the following design d. Each treatment is contained in r different blocks, while a collection of blocks (not necessary distinct) is derived from an initial block by successive addition of m to the labels in the initial block and reduction of t when necessary. Further, the treatment set can be partitioned into m mutually exclusive groups using residue classes modulo m. The residue class S_i ($i = 0, \ldots, m-1$) is formed by the collection of labels: $i, i+m, \ldots, i+m(n-1)$. The design d belongs to the class of generalized cyclic incomplete block designs (GCIBD for short). For basic properties of GCIBD see John (1987, Section 4.6) or Jarrett and Hall (1978).

The statistical analysis of interest is the intrablock analysis with the usual linear additive model postulated for incomplete block designs. Only connected designs are of interest here, as only these designs provide estimability of all treatment contrasts. As is customary, we assume that the linear model under considerations involves uncorrelated homoscedastic errors. Subsequently we shall drop σ^2 as a common variance of experimental errors.

Considerations given here rest on the notion of inter-contrasts, which arises from practical investigations into designing rotation experiments (see Section 4). Let us denote by $\tau' = (\tau_1, \ldots, \tau_t)$ the t-vector of treatment effects. This note focuses on estimation of the differences $\tau_p - \tau_q$ for p, q being from distinct modulo classes. These elementary contrasts have been termed inter-contrasts (see Bronowicka-Mielniczuk et al., 2000), although in slightly different design context. An examination of inter-contrasts allows to give a new characterization of GCIBD. Section 2 contains the main result of this paper. Section 3 provides numerical examples. A note on applications is given in Section 4.

Our interest in the subject stems from the practice concerning field trials on crop rotations. Cyclic block designs, being associated with a field set-up encountered in practice, appear to be useful in examination of its properties. The notion of inter-contrasts has been oriented towards statistical comparison of rotation designs.

2. Characterization of designs

Standard notation from linear algebra is used here. Given a matrix $\boldsymbol{A}$, the symbols $\boldsymbol{A}'$, tr$\boldsymbol{A}$, $\boldsymbol{A}^-$ denote, respectively, the transposition of $\boldsymbol{A}$, the trace of $\boldsymbol{A}$, and a generalized inverse of $\boldsymbol{A}$, i.e. $\boldsymbol{A}^-$ is such that $\boldsymbol{A}\boldsymbol{A}^-\boldsymbol{A} = \boldsymbol{A}$. The $n \times n$ identity matrix is denoted by $\boldsymbol{I}_n$, the $n \times n$ matrix with all entries equal to one is denoted by $\boldsymbol{J}_n$. In the sequel, we shall use $(\varphi_1, \ldots, \varphi_n)$, $(\varepsilon_1, \ldots, \varepsilon_m)$ to denote the standard basis of R^n and R^m, respectively. From now on, $\boldsymbol{\Pi}$ stands for the $n \times n$ matrix defined by the column partitioning $\boldsymbol{\Pi} = (\varphi_n, \varphi_1, \varphi_2, \ldots, \varphi_{n-1})$. The matrix $\boldsymbol{\Pi}$ is called the *basic circulant* of order n.

Let $\boldsymbol{C}$ denote the information matrix for estimation of $\boldsymbol{\tau}$ in d. Recall that $\boldsymbol{C}$ and $\boldsymbol{C}^-$ share the property of being a block circulant of type (n, m) (see John, 1987, p.78, Davis, 1994, Section 5.6). Thus

$$\boldsymbol{C} = \sum_{h=0}^{n-1} \boldsymbol{\Pi}^h \otimes \boldsymbol{C}_h \qquad \text{and} \qquad \boldsymbol{C}^- = \sum_{h=0}^{n-1} \boldsymbol{\Pi}^h \otimes \boldsymbol{\Theta}_h,$$

where $\boldsymbol{C}_h$, $\boldsymbol{\Theta}_h$ are square matrices of order m with elements c_{ij}^h and θ_{ij}^h, respectively.

To begin with, observe that the coefficient vectors associated to the inter-contrasts are defined by

$$\boldsymbol{t}_{(h,i,k,j)} = \varphi_h \otimes \varepsilon_i - \varphi_k \otimes \varepsilon_j,$$

where $1 \leq i < j \leq m$ and $1 \leq h, k \leq n$. Further

$$\varphi_h' \boldsymbol{\Pi}^a \varphi_k = \begin{cases} 1 & for\, a = k - h \\ 0 & otherwise \end{cases},$$

where $k - h$ is reduced modulo n. Hence

$$(\varphi_h \otimes \varepsilon_i)' \, C^- \, (\varphi_k \otimes \varepsilon_j) = \theta_{ij}^{k-h}. \tag{1}$$

If we let $V_\varkappa$ denote the total variance resulting from the estimates of inter-contrasts, then by (1)

$$\begin{aligned} V_\varkappa &= \sum_{1\le i<j\le m} \sum_{1\le h,k\le n} t'_{(h,i,k,j)} C^- t_{(h,i,k,j)} \\ &= \sum_{1\le i<j\le m} \sum_{1\le h,k\le n} \left(\theta_{ii}^0 + \theta_{jj}^0 - \theta_{ij}^{k-h} - \theta_{ji}^{k-h}\right) \\ &= n^2 (m-1) \operatorname{tr} \Theta_0 + n \sum_{k=0}^{n-1} \operatorname{tr} \Theta_k, \end{aligned} \tag{2}$$

the latter term being obtained on account of the property that row sums of the information matrix are equal to zero. For each choice of $1 \le i < j \le m$ put $u_{ij} = \left(\sum_{k=1}^{n} \varphi_k\right) \otimes (\varepsilon_i - \varepsilon_j)$. Applying the same reasoning as before we get

$$\sum_{1\le i<j\le m} u'_{ij} C^- u_{ij} = nm \sum_{k=0}^{n-1} \operatorname{tr} \Theta_k. \tag{3}$$

Further, direct calculations give

$$\sum_{h=0}^{n-1} C_h = r \left(I_m - m^{-1} J_m\right).$$

It follows immediately that u_{ij} are eigenvectors of C corresponding to the common eigenvalue r. This implies that $u'_{ij} C^- u_{ij} = 2nr^{-1}$. We conclude from (3) that

$$\sum_{k=0}^{n-1} \operatorname{tr} \Theta_k = (m-1) r^{-1}. \tag{4}$$

Denote by V the total variance resulted from the estimates of all pairwise comparisons. Since $V = tn \operatorname{tr} \Theta_0$ (see John, 1987, p.26), combining (2) with (4) yields

$$V_\varkappa = (m-1) m^{-1} V + n (m-1) r^{-1}.$$

Finally, set $V_\varkappa^M = \binom{m}{2}^{-1} n^{-2} V_\varkappa$, $V^M = \binom{t}{2}^{-1} V$ as the respective mean variances. The proceeding formula becomes

$$V^M - V_\varkappa^M = 2t^{-1} \left((t-1)^{-1} n \operatorname{tr} \Theta_0 - r^{-1}\right). \tag{5}$$

We shall utilise the following lemma due to Shah and Sinha (1989, p.70).

LEMMA 1. *Let a function f be convex and nonincreasing over $[0,\infty)$. Then for any connected design*

$$\sum_{i=1}^{t-1} f(\lambda_i) \geq \frac{t-1}{t} \sum_{i=1}^{t} f(\frac{t}{t-1} c_{ii}) ,$$

where $\lambda_1, ..., \lambda_{t-1}$ are nonzero eigenvalues of the information matrix, c_{ii} being its diagonal elements.

We summarise our findings in the following theorem.

THEOREM 1. *Let d be a generalized cyclic design with the pattern of construction given above. Then the mean variance resulted from the inter-contrasts does not exceed the overall mean variance.*

Proof. We only need to use *Lemma 1* by setting $f(x) = 1/x$. Then

$$n \operatorname{tr} \Theta_0 \geq \frac{(t-1)^2 k}{(k-1) tr}.$$

Hence, invoking (5), we obtain

$$V^M - V_{\varkappa}^M \geq 2t^{-1} r^{-1} \left(\frac{t-1}{t} \frac{k}{k-1} - 1 \right).$$

Condition $k \leq t$ completes the proof. □

3. Numerical Results

To explore numerically the properties discussed in Section 2 we considered some settings of practical importance. A summary of numerical findings is included in Table 1, which gives the results for $m = 2, 3$ and $n = 3, 4, 5$. The rightmost column presents the weak lower bound established in Theorem 2. For the sake of convenience in exposition, if a sequence of the same results follows, we leave empty cells with the exception of an initial item.

Table 1. Summary of the numerical comparisons

m	n	r	Initial block	V^M	$V^M_\varkappa$	β
2	3	2	(0...3)	1.1333	1.1111	.0185
2	4	2	(0...3)	1.2857	1.2500	.0208
			(0125)	1.2381	1.2083	
		3	(0...5)	.7024	.6979	.0042
2	5	2	(0...3)	1.4444	1.4000	.0200
			(0125)	1.2667	1.2400	
			(0127)			
		3	(0...5)	.7340	.7273	.0053
			(0...367)			
			(0...47)	.7259	.7200	
		4	(0...7)	.5148	.5133	.0014
3	3	2	(0...5)	1.0833	1.0740	.0074
3	4	2	(0...5)	1.1818	1.1667	.0083
			(0...378)	1.2182	1.2000	
			(0...48)	1.1364	1.1250	
		3	(0...8)	.6894	.6875	.0017
3	5	2	(0...5)	1.2857	1.2667	.0080
			(012678)			
			(0...48)	1.1818	1.1697	
			(0...378)			
		3	(0...8)	.7099	.7071	.0022
			(0...591011)			
			(0...711)	.7053	.7028	
			(0...61011)			
		4	(0...11)	.5095	.5089	.0006

4. Concluding remarks

This section is intended to motivate our investigations. To this end, we give a brief exposition of crop rotation designs.

Crop rotation is a sequence of crops grown consecutively on the same area of land. In rotations experiments species are alternated according to some fixed plans that take into account agronomic requirements. Many field studies in this area are designed to investigate variability in agricultural or biomass yields due to crop rotations as main management systems. The following features of experimental design should be noted. Suppose we are given m rotation schemes (also systems) enumerated $0, 1, \ldots, m-1$. Let n denote the number of components per rotation. As is customary, rotations are designed such that each system is to be applied n times, each with a different component as starting point for the rotation. Thus, each of the m rotation schemes contributes n sequences (treatments) to the design by successive shifting its elements cyclically to the right until the initial succession is achieved. It is relevant in the present context to enumerate the treatments as follows: treatments derived from the ith rotation are labelled by the successive elements of S_i, the ith residue group. To avoid difficulties with relative yield measures a common strategy employs so-called test crop, i.e. the experiment is performed by measuring a response for a selected item in rotations (see Przybysz, 1982, Bronowicka-Mielniczuk et al., 2000). Thus, n years are included into analysis and those years have incomplete data available. Let N be an $mn \times n$ $(0,1)$ matrix. If we label the rows of N by the treatment set and columns by years then 1 in the (i,j)th position means that there is the test species in the jth year on the plot with the ith sequence. With regard to the mentioned rules, the matrix N is a generalized circulant with the increment number equal to m (see John, 1980, Bronowicka-Mielniczuk and Mielniczuk, 2002). In view of comparative objectives, the inter-contrasts receive our particular attention. Consequently, it seems justified to judge quality of GCIBD by these contrasts. We close this note with brief mention that the results given here can be incorporated quite easily to non-additive models developed for analysing data from rotation experiments with a test species (see Bronowicka-Mielniczuk, 2002).

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O pewnej własności uogólnionych cyklicznych układów blokowych

STRESZCZENIE

W oparciu o koncepcję 'inter-kontrastów' obiektowych sformułowano nową charakterystykę uogólnionych układów cyklicznych. Rozważania obejmują przypadek jednakowych replikacji obiektowych. W uzupełnieniu rozważań teoretycznych omówiono w zarysie sens praktyczny podjętej problematyki.

SŁOWA KLUCZOWE: eksperyment płodozmianowy, inter-kontrast, macierz cykliczna, uogólniony cykliczny układ bloków niekompletnych